Is the Lorentz signature of the metric of spacetime electromagnetic in origin?

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Abstract

We formulate a premetric version of classical electrodynamics in terms of the excitation $H = (\mathcal{H}, \mathcal{D})$ and the field strength $F = (E, B)$. A local, linear, and symmetric spacetime relation between H and F is assumed. It yields, if electric/magnetic reciprocity is postulated, a *Lorentzian* metric of spacetime thereby excluding Euclidean signature (which is, nevertheless, discussed in some detail). Moreover, we determine the Dufay law (repulsion of like charges and attraction of opposite ones), the Lenz rule (the relative sign in Faraday's law), and the sign of the electromagnetic energy. In this way, we get a systematic understanding of the sign rules and the sign conventions in electrodynamics. The question in the title of the paper is answered affirmatively.

Key words: Metric of spacetime, classical electrodynamics, signature of metric, Lenz's rule, positivity of energy
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Running title

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1 Introduction

The spacetime structure presently used in physics is provided by special and by general relativity theory. Locally — and in special relativity also globally — one can introduce suitable coordinates $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, here c is the vacuum velocity of light, such that the metric reads

$$ds^2 = +c^2 dt^2 - dx^2 - dy^2 - dz^2\,. \tag{1}$$

The qualitative difference between time, with coordinate t, and space, with coordinates x, y, z, is reflected in the Minkowskian (also known as Lorentzian) $(+ - - -)$ signature of the metric of spacetime.

The form of the metric (1) raises immediately three questions that are related to one another: (i) Why is space 3-dimensional? (ii) Why is time 1-dimensional? (iii) Why does this specific choice of the signs appear in the signature?

Arguments have been provided in favor of the 3-dimensionality of space, such as the stability of orbits in a Newtonian potential, see Ehrenfest [5]. Also the sign difference $+ -$ in the signature has been addressed in different contexts. Greensite [8], for example, has put forward the idea to start with the complex signature $(e^{i\theta}, 1, ..., 1)$ and to treat θ as a quantum field that, under certain circumstances, can take the values $\theta = 0$ or $\pm\pi$. Thereby Euclidean or Lorentzian signatures could emerge, respectively. These ideas have been extended, see, e.g., Carlini and Greensite [3] and Odintsov et al. [18], by studying a quantum evolution equation and its consistency conditions. Tegmark [24], starting from superstring theories and applying arguments related to the anthropic principle, finds only the $1+3$ dimensional spacetime as habitable. Mankoč Borštnik

and Nielsen [16,17] have studied equations of motion for particles with spin in higher dimensional spaces. Requiring linearity in momentum, hermiticity, irreducibility under the Lorentz group, together with other technical assumptions, they could single out 1+1 and 1+3 dimensional spacetimes. This type of argument has been deepened by van Dam and Ng [4]. They consider 4 dimensions and base their work on Wigner's unitary irreducible representations of the (proper orthochronous) Poincaré group. Inquiring into the covering groups of the subgroups SO(4), SO(1,3), and SO(2,2) of the Poincaré type group, they find that a $0+4$ world has no interesting dynamics whereas a $2+2$ world can only have spin 0 particles; in contrast, a $1+3$ world has a rich dynamics.

Our approach is different. With exception of the Ehrenfest argument quoted, the other considerations are all of quantum (field) theoretical nature. Is this really plausible? The light cone $ds^2 = 0$ is defined physically by propagating classical electromagnetic waves ("light") in their geometrical optics limit. Accordingly, understanding the signature of the metric would appear to be equivalent to the understanding of the properties of propagating light in its geometrical optics limit (see [20]). In other words, if we somehow could deduce the light cone in the framework of *classical electrodynamics,* then the signature would come jointly with it, and in this way a relation could be established between the signature of the metric and the laws of electrodynamics. We will show, for the first time, that there is a relation between *four* signs in electrodynamics and the signature: the $-$ sign in the Ampère-Maxwell law $d\mathcal{H} \ominus \dot{\mathcal{D}} = j$, the $+$ sign in the Faraday law $dE \oplus \dot{B} = 0$ (the Lenz rule [1]), the $+$ sign of the electromagnetic energy density, and the $(+ - - -)$ signature of the Lorentz metric.

In the axiomatic *pre*metric approach to electrodynamics proposed recently [10,11,12], see also [6,19,21,25,26], the foundations of electrodynamics are formulated *before* a metric is introduced. Thus, different geometrical structures are introduced at different levels of the construction. This stratification of the structure of the theory gives a possibility to find out the relations between the sign assumptions mentioned above. The aim of this article is to establish such correlations of signs and even to derive the Lorentz signature of the underlying metric from electric/magnetic reciprocity.

The *pre*metric approach commences by assuming conservation of electric charge $\oint J = 0$ and of magnetic flux $\oint F = 0$ in n-dimensional spacetime. In order to attribute to these integral relations a proper physical meaning as conservation laws, we have to extract one dimension as 'topological time'. With this topological assumption, a formulation of Maxwell's theory in n dimensions is straightforward. Nevertheless, we will restrict the dimensions of the spacetime

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[1] Discussions of the physics of Lenz's rule can be found, e.g., in Jackson [14] — he calls it Lenz's law — or in Sommerfeld [22].

that we investigate. Our actual spacetime of 4 dimension is distinguished from other dimensions in that only for $n = 4$ the number of independent components of the electric field $n - 1$ equals to that of the magnetic field $(n-1)(n-2)/2$. In other words, the electromagnetic field strength $F = (E, B)$ as 2-form in n dimensions is only a 'middle form' for $n = 4$. This argument was already mentioned by Ehrenfest [5]. A 2-dimensional electron gas in the context of the quantum Hall effect, for example, can be considered by meaning of $(1 + 2)$-dimensional electrodynamics: Then, for $n = 3$, we have indeed 2 components of the electric field E but only 1 component of the magnetic field B. Such a 3-dimensional model of electrodynamics is basically different from the electrodynamics in $n = 4$. From now on we will assume $n = 4$.

Conventionally, in Maxwell's electrodynamics, a set of assumptions on signs are postulated partly a priori. A first assumption is that on the signs appearing in the Lorentz *metric*. A second one is *Lenz's rule* which establishes the sign in Faraday's induction law as opposed to the sign in Ampère–Maxwell's somewhat analogous law. And a third assumption is the customary set of signs in the *energy-momentum* tensor of the electromagnetic field. We would like to disentangle these interrelationships. It is well-known, for instance, that Lenz's rule can be derived from energy considerations. Since these assumptions on signs are postulated in Maxwell's electrodynamics all together 'at the same time', it seems to be impossible to find out relations between them. Earlier discussions of Maxwell's theory with Euclidean signature were given by Zampino [27] and on the Euclidean Maxwell-Einstein equations by Brill [2]. Both authors didn't use the premetric approach, even though Brill mentions it.

2 Premetric electrodynamics and its $(1 + 3)$-decomposition

2.1 Topological structure

The construction of pre-metric electrodynamics starts by postulating certain topological conditions on spacetime.

Axiom 1 *We require spacetime to be a 4-dimensional differential manifold X^4 that allows a smooth foliation of codimension 1. The foliation is denoted by a smoothly increasing parameter $\sigma \in (-\infty, \infty)$. Thus we have a partition of X^4 into disjoint path-connected subsets X^3_σ, which are local homeomorphic to $\mathbb{R}^3$. The parameter σ is the prototype of a time coordinate. We call σ the topological time.*

Fig. 1. Four-dimensional spacetime and a foliation of codimension 1 (see [12]). The coordinate x^3 is suppressed.
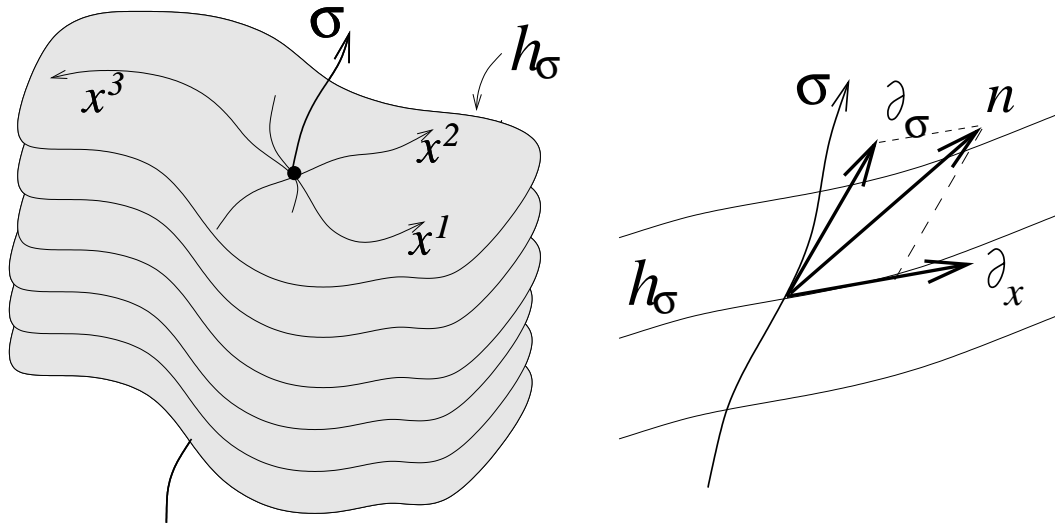
The manifold is considered without metric and without connection. The metric is derived in the theory from physical properties of the electromagnetic field. However, the restrictions on topology by Axiom 0 are essential for the existence of some physics on a topological manifold [7]. In particular, they are necessary for global hyperbolicity and for the existence of a spinorial structure.

With these topological assumptions, we are ready to formulate our fundamental structures. The electromagnetic quantities will be represented by differential forms of different degrees. For a given foliation σ, we are able to decompose them into *tangential* and *normal* pieces. An arbitrary p-form α may be decomposed uniquely (for a given σ) as

$$\alpha = \beta \wedge d\sigma + \gamma \,, \tag{2}$$

where the $(p-1)$-form β and the p-form γ are forms that lie in the folio, i.e., they satisfy the relations

$$e_0 \rfloor \beta = e_0 \rfloor \gamma = 0 \,, \qquad e_0 = \partial/\partial\sigma \,. \tag{3}$$

Observe that due to the covariance of the structures used, the forms β and/or γ cannot vanish identically. Indeed, if even one of them is zero for a given foliation σ, it will be non-vanishing for a foliation σ' that is only 'turned' by a small amount.

We define the positive volume element in X^4 as

$$^{(4)}\text{vol} = d\sigma \wedge {}^{(3)}\text{vol}\,, \tag{4}$$

where $^{(3)}\text{vol}$ is a positive volume element on a hypersurface of constant σ. The 4-dimensional exterior derivative decomposes as

$$d = d\sigma \wedge \frac{\partial}{\partial\sigma} + \underline{d}\,, \tag{5}$$

where $\underline{d}$ refers to the local coordinates lying in the hypersurface of constant σ. The partial derivative $\partial/\partial\sigma$ is abbreviated by a dot on top of the corresponding quantity.

A 3-form α that lies in the folio has specific properties. Because of antisymmetry, the space-like exterior derivative of such a form is zero: $\underline{d}\,\alpha = 0$. Moreover, an arbitrary α is proportional to the volume element $^{(3)}\text{vol}$. Hence we can distinguish positive and negative 3-forms that are transversal to the folio.

2.2 Continuity equation for electric charge

Let us now postulate the existence of an electric charge current density.

Axiom 2 *The charge current density is a conserved twisted 3-form J, i.e., the 3-form J, being integrated over an arbitrary closed 3D submanifold $C_3 \in X^4$, obeys*

$$\oint_{C_3} J = 0\,, \quad \partial C_3 = 0 \qquad \Longrightarrow \qquad dJ = 0\,. \tag{6}$$

The decomposition of J into tangential and normal pieces relative to the foliation σ may be written as

$$J = i_{\mathrm{T}}\, j \wedge d\sigma + i_{\mathrm{s}}\, \rho\,, \tag{7}$$

where we introduced the **T**ime and **S**pace factors $i_{\mathrm{T}}, i_{\mathrm{s}}$ with values from the set $\{+1, -1\}$. Any factor the absolute value of which is different from 1 is considered to be absorbed in the corresponding form. Hence (7) is the most general decomposition relative to the given foliation σ. We are going to derive which values of the factors $i_{\mathrm{T}}, i_{\mathrm{s}}$ are in correspondence with the physical interpretation of the forms j and ρ.

Fig. 2. Charge conservation in 4-dimensional space (see [12]). Here $t = \sigma$ denotes the topological time.
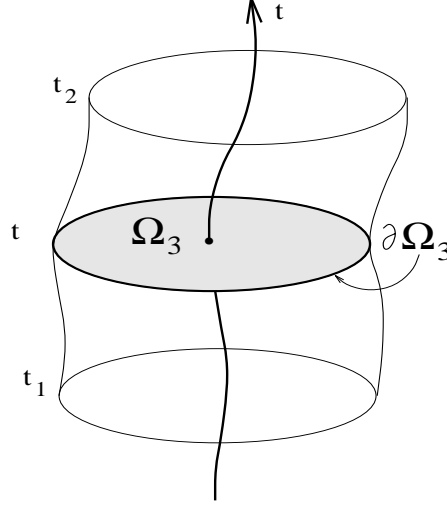
We differentiate (7) and use $\underline{d}\,\rho = 0$:

$$d\,J = i_{\mathsf{T}}\,d\,j \wedge d\sigma + i_{\mathsf{S}}\,d\,\rho = d\sigma \wedge (-i_{\mathsf{T}}\,\underline{d}\,j + i_{\mathsf{S}}\,\dot\rho)\,. \tag{8}$$

Accordingly, $d\,J = 0$ yields

$$\underline{d}\,j - \frac{i_{\mathsf{S}}}{i_{\mathsf{T}}}\,\dot\rho = 0\,. \tag{9}$$

Let us now take into account the relation between the forms j and ρ. The 3-form ρ represents the space density of charge, whereas the 2-form j represents the current density of the *same* charge. Then (9) has to represent the differential $(1+3)$-dimensional expression of the conservation law of electric charge. Consider the integral version of (9). For this, we have to choose a compact $3D$ region Ω_3 with the $2D$ boundary $\partial\Omega_3$ and a normal vector field n directed outwards of $\partial\Omega_3$, i.e., we assume that the usual conventions are valid. Then, integrating (9) over the region Ω_3,

$$\int\limits_{\Omega_3} \left(\underline{d}\,j - \frac{i_{\mathsf{S}}}{i_{\mathsf{T}}}\,\dot\rho \right) = 0\,, \tag{10}$$

or by using the divergence theorem,

$$\int\limits_{\partial\Omega_3} (n\rfloor j)dS = \frac{i_{\mathsf{S}}}{i_{\mathsf{T}}}\,\frac{\partial}{\partial\sigma} \int\limits_{\Omega_3} \rho\,. \tag{11}$$

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The left hand side of (11) represents the flux of charge through the boundary $\partial\Omega_3$. Let the σ axis be directed in the future. Then, in the case of a positive flux, the change of the flux in the compact domain bound by $\partial\Omega_3$ has to be negative.

Thus, the right hand side has to represent the change of the charge in the compact domain bounded by $\partial\Omega_3$. Consequently,

$$i_\mathsf{S} = -i_\mathsf{T}\,. \tag{12}$$

In this case, eq.(9) turn out to be the continuity equation

$$\underline{d}\,j + \dot{\rho} = 0\,, \tag{13}$$

which represents the appropriate conservation law.

A pointwise charge density, for example,

$$\rho = e\,\delta^{(3)}(\vec{r} - \vec{r}_0)\,dx \wedge dy \wedge dz\,, \tag{14}$$

provided we define the current in the conventional way by

$$j = v\rfloor\rho\,, \tag{15}$$

fulfills (9) only for the parameters (12). Indeed, differentiating (14), we obtain

$$\frac{\partial\rho}{\partial\sigma} = \frac{\partial\rho}{\partial x_0^i}\frac{\partial x_0^i}{\partial\sigma} = -\frac{\partial\rho}{\partial x^i}v^i = -d(v\rfloor\rho) = -dj\,. \tag{16}$$

Consequently, by (12), the true decomposition of the 4-dimensional current J, with the convention (15), reads

$$J = i_\mathsf{T}\,(j \wedge d\sigma - \rho)\,. \tag{17}$$

The "$-$" sign in this decomposition originates, via (8), from the odd degree of the form $\underline{d}\,j$. Nevertheless, the continuity equation (13) involves the "$+$" sign. This decomposition can be also considered as a choice of the proper direction of the σ-axis.

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2.3 Inhomogeneous Maxwell equation

By de Rham's theorem, the inhomogeneous Maxwell equation is a consequence of the charge conservation law $(6)_1$,

$$J = dH \,, \tag{18}$$

where H is the twisted 2-form of the *electromagnetic excitation*, which has the absolute dimension of charge. In the $(1+3)$-decomposition of the excitation

$$H = h_{\mathrm{T}} \, \mathcal{H} \wedge d\sigma + h_{\mathrm{S}} \, \mathcal{D} \,, \tag{19}$$

we introduced again the sign factors h_{T} and h_{S} with values from the set $\{+1, -1\}$. We differentiate (19)

$$
\begin{aligned}
dH &= h_{\mathrm{T}} \, \underline{d}\mathcal{H} \wedge d\sigma + h_{\mathrm{S}} \, \underline{d}\mathcal{D} + d\sigma \wedge \dot{\mathcal{D}} \\
&= d\sigma \wedge \left(h_{\mathrm{T}} \, \underline{d}\mathcal{H} + h_{\mathrm{S}} \, \dot{\mathcal{D}} \right) + h_{\mathrm{S}} \, \underline{d}\mathcal{D} \,.
\end{aligned} \tag{20}
$$

Thus the inhomogeneous Maxwell equation (18), with the source (17), decomposes according to

$$h_{\mathrm{T}} \, \underline{d}\mathcal{H} + h_{\mathrm{S}} \, \dot{\mathcal{D}} = i_{\mathrm{T}} j \,, \qquad h_{\mathrm{S}} \, \underline{d}\mathcal{D} = -i_{\mathrm{T}} \rho \,. \tag{21}$$

2.4 Lorentz force

This axiom introduces implicitly the *electromagnetic field strength F* as an independent concept via the mechanical concept of force and the existence of prescribed electric test currents.

Axiom 3 *The Lorentz force density (a twisted covector-valued 4-form) is postulated as*

$$f_\alpha = (e_\alpha \rfloor F) \wedge J \,, \tag{22}$$

where e_α is the frame and F an untwisted 2-form.

The field strength F has the absolute dimension of magnetic flux. Its $(1+3)$-decomposition reads

$$F = f_{\mathrm{T}} \, E \wedge d\sigma + f_{\mathrm{S}} \, B \,, \tag{23}$$

where the sign factors f_T and f_S take values from $\{+1, -1\}$.

Thus, the Lorentz force density (22) decomposes according to

$$f_0 = -i_T f_T\, E \wedge j \wedge d\sigma \tag{24}$$

and

$$f_\mu = \left[i_T f_T (e_\mu \rfloor E)\rho + i_T f_S (e_\mu \rfloor B) \wedge j \right] \wedge d\sigma\,, \qquad \mu = 1, 2, 3\,. \tag{25}$$

2.5   Homogeneous Maxwell equation

From a formal point of view, we have four equations (22) for six components of the 2-form F. Additional condition have to be employed in order to fix them uniquely.

Axiom 4 *The conservation of magnetic flux is postulated; as a consequence, we find the homogeneous Maxwell equation,*

$$\oint_{C_2} F = 0\,, \quad \partial C_2 = 0\,, \qquad \Longrightarrow \qquad dF = 0\,, \tag{26}$$

for any closed submanifold C_2.

By de Rham's theorem, $(26)_1$ yields

$$F = dA\,, \tag{27}$$

where A is the untwisted 1-form of *the electromagnetic potential.* According to (26), the field strength F would be only determined up to the differential of an arbitrary 1-form. However, the expression for the Lorentz force (22) defines it uniquely. In accordance with (23), the homogeneous Maxwell equation $dF = 0$ decomposes as

$$\begin{aligned} dF &= f_T\, \underline{d}\, E \wedge d\sigma + f_S \left(\underline{d}\, B + d\sigma \wedge \dot{B} \right) \\ &= d\sigma \wedge \left(f_T\, \underline{d}\, E + f_S\, \dot{B} \right) + f_S\, \underline{d}\, B = 0\,. \end{aligned} \tag{28}$$

Thus,

$$f_T\, \underline{d}\, E + f_S \dot{B} = 0\,, \qquad \underline{d}\, B = 0\,. \tag{29}$$

3 Energy-momentum current

3.1 Energy-momentum current and Lorentz force

Up to now, we introduced the electromagnetic field (H, F) and its field equations $dH = J$ and $dF = 0$. At this stage of our construction, no specific geometric structure is provided. We still deal with a (topological) differential manifold without metric and without connection. The energy-momentum content of the electromagnetic field is described by means of a covector valued 3-form. We will denote it by Σ_α, with $\alpha = 0, 1, 2, 3$. It should be constructed in a covariant way in terms of the fields H and F and the frame field e_α. Certainly, for a system involving the electromagnetic field together with a source J, the current Σ_α of the electromagnetic field alone cannot be conserved. Moreover, the Lorentz force density f_α has to be treated as a *source* of the energy-momentum current, or, conversely, Σ_α is a kind of generalized *potential* for f_α. Since $J = dH$, the Lorentz force contains the derivative of the field H. For the energy-momentum current we assume that it depends only on the fields, but not on their derivatives.

Consider the 4-form $d\Sigma_\alpha$. Due to the linearity of the exterior derivative operator, this quantity can be represented as a sum of two terms: $d\Sigma_\alpha = \mathfrak{f}_\alpha + X_\alpha$. In the first term $\mathfrak{f}_\alpha$ the derivatives of the electromagnetic field are involved, whereas in the second term X_α there occur the derivatives of the frame field. The first term describes how the energy-momentum current changes under a temporal and spatial variation of the electromagnetic field. This is exactly what the Lorentz force density is supposed to mean. Thus, we have to assume $\mathfrak{f}_\alpha = f_\alpha$ and, consequently,

$$d\Sigma_\alpha - f_\alpha = X_\alpha \,. \tag{30}$$

Recall that the term X_α in (30) does *not* involve derivatives of F and H.

We substitute the expression for the Lorentz force $f_\alpha = (e_\alpha \rfloor F) \wedge J$ in (30) and use the inhomogeneous Maxwell equation $J = dH$. Moreover, we may add an arbitrary term proportional to the left-hand-side of the homogeneous Maxwell equation $dF = 0$:

$$d\Sigma_\alpha = (e_\alpha \rfloor F) \wedge dH + c\,(e_\alpha \rfloor H) \wedge dF + X_\alpha \,. \tag{31}$$

The right hand side already involves the exterior differential operator d. Thus, the current Σ_α has to be linear in H and F and no derivatives can occur.

Accordingly, we are led to the expression

$$\Sigma_\alpha = a(e_\alpha \rfloor F) \wedge H + b(e_\alpha \rfloor H) \wedge F\,, \tag{32}$$

with undefined constants a and b.

Thus we conclude that the energy-momentum current Σ_α has to be described by a covector-valued 3-form bilinear in the fields H and F and that it is physically meaningful only for definite values of the constants a and b. Moreover, the values of the constants a and b have to guarantee the compatibility with the first four axioms. As a covector-valued 3-form, Σ_α has in general $4 \times 4 = 16$ independent components.

Although (30) is covariant under local frame transformations, its left and right hand sides, respectively, are certainly not covariant themselves. However, this separation is meaningful in the following sense: Due to the linearity of the derivative operator d (Leibniz rule), an arbitrary local change of the frame only yields additional terms that are proportional to the fields H and F but not to their derivatives dH and dF, see [13] for details. Indeed, under a linear transformation of the frame, $e_{\alpha'} = L_{\alpha'}{}^\beta e_\beta$, the (covariant) energy-momentum current transforms as $\Sigma_{\alpha'} = L_{\alpha'}{}^\beta \Sigma_\beta$ and the Lorentz force as $f_{\alpha'} = L_{\alpha'}{}^\beta f_\beta$. We multiply (30) by $L_{\alpha'}{}^\alpha$. Then

$$d\left(L_{\alpha'}{}^\beta \Sigma_\beta\right) - dL_{\alpha'}{}^\beta \wedge \Sigma_\beta - L_{\alpha'}{}^\beta f_\beta = L_{\alpha'}{}^\beta X_\beta \tag{33}$$

or

$$d\Sigma_{\alpha'} - f_{\alpha'} = L_{\alpha'}{}^\beta X_\beta + dL_{\alpha'}{}^\beta \wedge \Sigma_\beta =: X_{\alpha'}\,. \tag{34}$$

Thus, although the separate terms in (30) are not covariant, the property of the left hand side to contain the derivatives of the electromagnetic field (F, H) and of the right hand side to contain the derivative of the frame field e_α is preserved under an arbitrary local linear transformation of the frame.

Accordingly, in (30), dH and dF appear only on the left hand side and this is preserved under local linear frame transformations. Consequently, the numerical values of the coefficients a and b of the energy-momentum current (32) have to be chosen such that in (30) the derivative terms dH and dF on its left hand side cancel each other. Substituting (32) in (30) we obtain

$$a\, d(e_\alpha \rfloor F) \wedge H - a(e_\alpha \rfloor F) \wedge dH + b\, d(e_\alpha \rfloor H) \wedge F - f_\alpha = X_\alpha \tag{35}$$

or, equivalently,

$$(a\,\mathcal{L}_\alpha F \wedge H + b\,\mathcal{L}_\alpha H \wedge F) - (a - b + 1)f_\alpha = X_\alpha \,, \tag{36}$$

where we introduced a short-hand notation for the Lie derivative operator $\mathcal{L}_\alpha := \mathcal{L}_{e_\alpha} = d(e_\alpha\rfloor) + e_\alpha\rfloor d$.

The terms with dH and dF in the first two terms of (36) have to vanish independently of the third term. Indeed, they have to vanish even in the sourcefree case when the Lorentz force is zero. Hence,

$$a - b + 1 = 0 \,. \tag{37}$$

Eq.(36) now simplifies to

$$a\,\mathcal{L}_\alpha F \wedge H + b\,\mathcal{L}_\alpha H \wedge F = X_\alpha \,. \tag{38}$$

So far, the fields H and F are completely independent. The derivatives on the left hand side cannot cancel one another for any nonzero values of the parameters a and b. This situation is not surprising. Indeed, the electromagnetic field (H, F) has 12 independent components, whereas it is governed by the 8 independent field equations (18) and (26). Thus the system is underdetermined and it is natural that it does not provide a uniquely defined energy-momentum current. We postpone the treatment of (38) to the next section where a relation between the fields H and F will be introduced.

Our intermediate result is then, see (37), that the energy-momentum current reads

$$\Sigma_\alpha = a(e_\alpha\rfloor F) \wedge H + (a + 1)(e_\alpha\rfloor H) \wedge F \,. \tag{39}$$

If we transvect (39) with the coframe ϑ^α, we find the so-called *trace* of the energy-momentum current

$$\vartheta^\alpha \wedge \Sigma_\alpha = 2(2a + 1)F \wedge H \,. \tag{40}$$

This expression is a 4-form and has 1 independent component. It is equivalent to an irreducible piece of Σ_α. The tracefree piece of the energy-momentum current with 15 independent components can be defined as

$$\slashed{\Sigma}_\alpha := \Sigma_\alpha - \frac{1}{4}e_\alpha\rfloor\left(\vartheta^\beta \wedge \Sigma_\beta\right) \,. \tag{41}$$

3.2 Axiom on energy-momentum

From field theory we know, see [15], that for a non-scalar field the trace of the energy-momentum tensor is always related to the mass of the corresponding field. Thus, in pre-metric electrodynamics, the electromagnetic field (the "photon" field) is massless provided $\vartheta^\beta \wedge \Sigma_\beta$ vanishes or, equivalently, $a = -1/2$. This relation will be justified in an alternative way (without referring to the trace) in the next section. Still, we can already now formulate our axiom:

Axiom 5 *The energy-momentum current of the electromagnetic field is the covector-valued 3-form*

$$\Sigma_\alpha := \frac{1}{2}\left[(e_\alpha \rfloor H) \wedge F - (e_\alpha \rfloor F) \wedge H\right]. \tag{42}$$

Accordingly, $\vartheta^\alpha \wedge \Sigma_\alpha = 0$, i.e., our energy-momentum current is traceless $\not\Sigma_\alpha = \Sigma_\alpha$ and carries 15 independent components.

Let us now decompose this current into time and space components. Since we are interested in the *energy* of the electromagnetic field, it is sufficient to discuss the "time" ($t = \sigma$) component Σ_0. Because of $e_0 \rfloor d\sigma = 1$ and $e_0 \rfloor \mathcal{H} = e_0 \rfloor \mathcal{D} = e_0 \rfloor E = e_0 \rfloor B = 0$ (forms lie in the folio σ), we find $e_0 \rfloor H = -h_{\mathrm{T}} \mathcal{H}$ and $e_0 \rfloor F = -f_{\mathrm{T}} E$. Thus, the decomposition of Σ_0 turns out to be

$$\Sigma_0 = \frac{1}{2} f_{\mathrm{T}} E \wedge (h_{\mathrm{T}} \mathcal{H} \wedge d\sigma + h_{\mathrm{s}} \mathcal{D}) - \frac{1}{2} h_{\mathrm{T}} \mathcal{H} \wedge (f_{\mathrm{T}} E \wedge d\sigma + f_{\mathrm{s}} B)$$

$$= h_{\mathrm{T}} f_{\mathrm{T}} E \wedge \mathcal{H} \wedge d\sigma + \frac{1}{2} h_{\mathrm{s}} f_{\mathrm{T}} \mathcal{D} \wedge E - \frac{1}{2} h_{\mathrm{T}} f_{\mathrm{s}} \mathcal{H} \wedge B. \tag{43}$$

The relation between the sign of the Lorentz force and the sign of the energy-momentum current can be read off from

$$d\Sigma_0 = f_0 + X_0, \tag{44}$$

see (30).

3.3 Reciprocity symmetry

In the absence of a source J, the Maxwell field equations $dH = 0$ and $dF = 0$ do not change under the transformation $H \rightarrow \mu F$ and $F \rightarrow \nu H$. Here μ and ν are non-vanishing dimensionful *constants*: $\mu \neq 0$ and $\nu \neq 0$. This symmetry is expected to be preserved in the energy-momentum current Σ_α

of the electromagnetic field since this current has to be independent of the source. A look at (42) shows that this transformation yields

$$\Sigma_\alpha \rightarrow \frac{\mu\nu}{2}\left[-(e_\alpha\rfloor H)\wedge F + (e_\alpha\rfloor F)\wedge H\right] = -\mu\nu\Sigma_\alpha\,. \qquad (45)$$

Consequently,

$$\mu\nu = -1 \qquad (46)$$

yields the invariance $\Sigma_\alpha \rightarrow \Sigma_\alpha$. Hence Σ_α is only invariant under the transformation $H \rightarrow \mu F$, $F \rightarrow -H/\mu$.

In this analysis of Σ_α we used the symmetry $H \rightarrow \mu F$ and $F \rightarrow -H/\mu$ of the sourcefree Maxwell equations. However, as a rule, one should preferably investigate a symmetry in Lagrangians or Hamiltonians (i.e., energy-momentum expressions), see a corresponding remark of Staruszkiewicz [23]. Therefore we sharpen our notions and recognize that the energy-momentum current (42) (not, however, the Maxwell equations) is invariant under the transformation

$$H \rightarrow \zeta F\,, \quad F \rightarrow -\frac{1}{\zeta}H\,. \qquad (47)$$

We refer to it as *electric/magnetic reciprocity*. The function $\zeta = \zeta(x)$ is an arbitrary twisted 0-form (pseudo-scalar function) of dimension $[\zeta] = [H]/[F] = 1/\text{resistance}$. Because of $H = h_{\text{T}}\mathcal{H} \wedge d\sigma + h_{\text{s}}\mathcal{D}$ and $F = f_{\text{T}}E \wedge d\sigma + f_{\text{s}}B$, the $(1+3)$-decomposition of the reciprocity transformation (47) reads

$$\left\{\mathcal{H} \rightarrow \zeta\frac{f_{\text{T}}}{h_{\text{T}}}E\,, \quad \mathcal{D} \rightarrow \zeta\frac{f_{\text{s}}}{h_{\text{s}}}B\right\}\,, \qquad \left\{E \rightarrow -\frac{1}{\zeta}\frac{h_{\text{T}}}{f_{\text{T}}}\mathcal{H}\,, \quad B \rightarrow -\frac{1}{\zeta}\frac{h_{\text{s}}}{f_{\text{s}}}\mathcal{D}\right\}\,. \qquad (48)$$

Accordingly, electric quantities are replaced by magnetic ones and vice versa.

4   Spacetime relation

4.1   Constitutive tensor of spacetime

So far, the electromagnetic field (H, F) is underdetermined. It has 12 independent components that satisfy only 8 independent equations $dH = J$ and $dF = 0$. We need a *spacetime relation* linking the excitation H to the field strength F. First of all, we require that this "constitutive law for vacuum" is

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local, that is, H at a certain event with coordinates $x = (x^0, x^1, x^2, x^3)$ depends only on F at the same event. Neither differentials nor integrals are allowed. With a local operator $\kappa = \kappa(x)$ we can write $H = \kappa(F)$. Here κ is a twisted operator that does not depend on H and F. Furthermore, we require *linearity* of this operator. With arbitrary constants a and b, we have

$$H = \kappa(F), \qquad \kappa(a\,\Phi + b\,\Psi) = a\,\kappa(\Phi) + b\,\kappa(\Psi),\tag{49}$$

for arbitrary 2-forms Φ and Ψ.

We decompose H and F into their components

$$H = \frac{1}{2}H_{\alpha\beta}\,\vartheta^\alpha \wedge \vartheta^\beta, \quad F = \frac{1}{2}F_{\alpha\beta}\,\vartheta^\alpha \wedge \vartheta^\beta.\tag{50}$$

Here ϑ^α denotes the coframe. Using linearity, we can represent the linear operator κ componentwise:

$$H = \kappa(F) \quad \Longleftrightarrow \quad H_{\alpha\beta} = \frac{1}{2}\kappa_{\alpha\beta}{}^{\gamma\delta}F_{\gamma\delta}.\tag{51}$$

For a compact representation of the *constitutive tensor* $\kappa_{\alpha\beta}{}^{\gamma\delta}$ of spacetime (of the "vacuum"), it is useful to apply a formalism with indices running from 1 to 6: $I, J, \ldots = \{1, 2, 3, 4, 5, 6\} = \{01, 02, 03, 23, 31, 12\}$. The constitutive tensor takes now the form of a 6D-tensor $\kappa_I{}^J$ with 36 independent components. Two other important quantities, which also possess a 6×6 representation, are the Levi-Civita symbols ϵ^{IJ} and ϵ_{IJ}. These symmetric tensor densities play the role of a quasi-metric in the 6-dimensional vector space of 2-forms. In particular, they can be used for the raising and lowering of the indices of a tensor. Nevertheless, this procedure has to be carefully distinguished from the usual raising and lowering of indices by means of a 4D metric. After all, we are dealing here with *premetric* electrodynamics.

In this way we define the tensor density

$$\chi^{IJ} := \epsilon^{IM}\kappa_M{}^J, \qquad \kappa_I{}^J = \epsilon_{IM}\,\chi^{MJ}.\tag{52}$$

The constitutive tensor, in the form χ^{IJ} as 6×6 matrix, can be straightforwardly decomposed into its irreducible parts, the traceless symmetric part, the antisymmetric part, and the trace:

$$\chi^{IJ} = {}^{(1)}\chi^{IJ} + {}^{(2)}\chi^{IJ} + {}^{(3)}\chi^{IJ}.\tag{53}$$

The *principal* part ${}^{(1)}\chi^{IJ}$ has 20 independent components, the *skewon* part ${}^{(2)}\chi^{IJ}$ 15 components, and the *axion* part ${}^{(3)}\chi^{IJ} = \epsilon^{IJ}\alpha$ is equivalent to one

pseudo-scalar field α.

4.2 Energy-momentum current once more

Now we are able to go back to (38):

$$a \, \mathcal{L}_\alpha F \wedge H + b \, \mathcal{L}_\alpha H \wedge F = X_\alpha \, . \tag{54}$$

We assume the local and linear spacetime relation (51). We use the linearity of the Lie derivative and find

$$\mathcal{L}_\alpha H \wedge F = \mathcal{L}_\alpha \kappa(F) \wedge F = \kappa(\mathcal{L}_\alpha F) \wedge F + Y_\alpha \, . \tag{55}$$

Here Y_α denotes those terms that are obtained by taking the Lie derivative of the operator κ. Thus Y_α does not involve dH and dF.

In order to proceed, we need an another property of κ. We assume that in (53) the skewon piece vanishes. Then χ is *symmetric* and

$$\kappa(\Phi) \wedge \Psi = \Phi \wedge \kappa(\Psi) \, , \tag{56}$$

for any 2-forms Φ and Ψ. Since $\mathcal{L}_\alpha F$ is a 2-form, (55) can be rewritten as

$$\mathcal{L}_\alpha H \wedge F = \mathcal{L}_\alpha F \wedge H + Y_\alpha \, . \tag{57}$$

We substitute this in (54) and find

$$(a + b) \, \mathcal{L}_\alpha F \wedge H = X_\alpha - bY_\alpha \, . \tag{58}$$

Recall that only in the left hand side of this equation the derivative of the field F is involved. Thus, in addition to (37), we have a second relation between the coefficients, namely

$$a + b = 0 \, . \tag{59}$$

Consequently,

$$a = -\frac{1}{2} \, , \qquad b = \frac{1}{2} \, , \tag{60}$$

and, finally, we recover the energy-momentum current of Axiom 4. Observe that in our analysis we applied Maxwell's field equations and the expression

for the Lorentz force, that is, Axioms 1, 2, and 3, as well as *locality, linearity, and symmetry* of the spacetime relation. However, neither a specific metric nor a connection have been used.

A word of caution is in order: Our Axiom 5 is logically independent of the spacetime relation. As soon as H and F are specified, an energy-momentum current à la Axiom 5 always exists. If we assume additionally a local, linear, and symmetric spacetime relation, then $f_\alpha = d\Sigma_\alpha +$ (terms depending only on H and F). Thus a specific spacetime relation implies a specific form of the equation that is related to energy-momentum conservation.

4.3 Electric/magnetic reciprocity of the spacetime relation

Electric/magnetic reciprocity means that a specific exchange of the fields H and F preserves the energy-momentum current Σ_α. This is possible since Σ_α is algebraically expressed in terms of the fields H and F. Accordingly, it is natural to assume that electric/magnetic reciprocity is also applicable to another algebraic equation — the spacetime relation (51). Hence, with (47) and $(49)_2$, we obtain

$$H = \kappa(F) \qquad \Longrightarrow \qquad \zeta F = -\frac{1}{\zeta}\kappa(H) = -\frac{1}{\zeta}\kappa^2(F)\,. \tag{61}$$

Since F is arbitrary, κ^2 turns out to be proportional to the identity operator of the 6D vector space:

$$\kappa^2 = -\zeta^2\,\mathbb{I}_6\,. \tag{62}$$

The energy-momentum current was electric/magnetic reciprocal for arbitrary ζ. Not so for the spacetime relation. The components of the linear operator κ are directly observable. Thus κ^2 must not depend on an arbitrary function ζ^2. For this reason, we take the trace of (62) and resolve it with respect to ζ^2:

$$\zeta^2 = -\frac{1}{6}\operatorname{Tr}(\kappa^2) = -\frac{1}{6}\kappa_I{}^K\kappa_K{}^M\,. \tag{63}$$

Accordingly, ζ is no longer an arbitrary function, it is rather expressed in terms of the constitutive tensor κ.

The square of a real *pseudo*-scalar field is a real scalar field. Hence, instead of the pseudo-scalar field ζ, we may introduce a "true" scalar field λ by means

of the relation $\lambda^2 := \zeta^2$. The physical dimension of this new scalar field is $[\lambda] = 1/\text{resistance}$. Accordingly,

$$\kappa^2 = -\lambda^2 \, \mathbb{I}_6 \, . \tag{64}$$

Therefore, we are able to introduce an almost complex structure on the 6D vector space

$$\mathbb{J} := \frac{1}{\lambda} \, \kappa \, , \qquad \mathbb{J} = \sqrt{-\mathbb{I}_6} \, . \tag{65}$$

4.4 Signature emerges

The principal part of the constitutive tensor obeys the symmetry $\chi^{IJ} = \chi^{JI}$. Let us put the other two other parts to zero, namely the skewon and the axion parts. Consequently, for arbitrary 2-forms Φ and Ψ, we have the symmetry

$$\kappa(\Phi) \wedge \Psi = \kappa(\Psi) \wedge \Phi \, , \qquad \mathbb{J}(\Phi) \wedge \Psi = \mathbb{J}(\Psi) \wedge \Phi \, . \tag{66}$$

Moreover,

$$\mathbb{J}^2 \Phi = -\Phi \, . \tag{67}$$

In such a way we have constructed a local and linear operator $\mathbb{J}$ that is (i) symmetric, (ii) maps twisted 2-forms to untwisted ones, and (iii), if squared, equals to the negative of the identity operator. These properties, here found for an operator acting on 2-forms, are the those of the Hodge star operator. Therefore, our operator $\mathbb{J}$ corresponds to the Hodge star operator $*_{(g)}$, constructed (uniquely) from some metric g given on the manifold. The square of the Hodge operator, acting on a p-form ω, is given by

$$*^2 \, \omega = (-1)^{p(n-p)+\text{ind}} \, \omega \, , \tag{68}$$

where n is the dimension of the manifold and ind the index of the metric (the number of the minus signs in the signature). For 2-forms Φ on a $4D$-manifold, this formula yields

$$*^2 \, \Phi = (-1)^{\text{ind}} \, \Phi \, . \tag{69}$$

Comparing this relation with (67) we derive

$$\text{ind} = 1, \, 3 \, . \tag{70}$$

The unique signature of such indices on a $4D$-manifold is Lorentzian $(-1, +1, +1, +1)$ or, equivalently, $(+1, -1, -1, -1)$.

Thus we obtain the important result: *The electric/magnetic reciprocity of the energy-momentum current, if applied to a local, linear, and symmetric spacetime relation, yields an operator that is equivalent to the Hodge operator of a metric with Lorentzian signature.* Accordingly, the answer to the question posed in the title of our paper is clearly affirmative.

4.5 Maxwell-Lorentz spacetime relation

So far, we considered electrodynamics (field equations and conservation laws) on a metric-free background. We have shown that a metric of Lorentzian signature is singled out by its correspondence to a specific symmetry requirement of the spacetime relation, namely to its electric/magnetic reciprocity (see Sec.4.3). Let us consider now a manifold endowed with a metric g of a certain signature. Our goal is twofold. On the one hand, we want to establish which of the sign factors of electrodynamics is induced by the Lorentzian signature. On the other hand, we want to examine which sign factors and, correspondingly, which laws of electrodynamics emerge in the case of a Euclidean metric. The last question was discussed by Zampino [27] and Brill [2]. Since sign factors as well as the signature of the metric do not depend on a point, it is enough to deal with a local metric g referred to orthogonal axes. For our foliated manifold it means that we should take orthonormal frames in a folio and a σ-axis normal to the folio. Thus, the components of the metric read

$$g_{\alpha\beta} = \mathrm{diag}\Big((-1)^{s_0}\, c^2, (-1)^{s_1}, +1, +1\Big), \qquad s_0, s_1 \in \{0, 1\}, \tag{71}$$

with c as velocity of light. This expression for the components of the metric embodies all possible signatures, see Table 1.

Table 1. Signature and the exponents s_0, s_1 in Eq.(71)

Signature	s_0	s_1
Minkowski (aka Lorentz)	1	0
Euclid	0	0
$(-, -, +, +)$	1	1

We specialize the constitutive tensor by demanding vanishing of the skewon and axion pieces. These two additional fields to ordinary electrodynamics do

not affect the sign factors. The spacetime relation takes now the standard form

$$H = \lambda * F \,, \tag{72}$$

wherein the Hodge operator $*$ is defined in terms of the metric (71). The scalar function λ, a dilaton type field, can also be considered as an addendum to standard electrodynamics. By requiring λ to be constant, we discard also this field.

For the $(1+3)$ decomposition of (72) we introduce the 3-dimensional Hodge operator $\underline{*}$. For the diagonal metric (71), the interrelation between $*$ and $\underline{*}$ takes a rather simple form,

$$* (d\sigma \wedge E) = (-1)^{s_0} \frac{1}{c} \underline{*} E \,, \qquad *B = c \, d\sigma \wedge \underline{*} B \,, \tag{73}$$

see [12]. Because of (68), we have $\underline{*}^2 = (-1)^{s_1}$. Thus the $(1+3)$ decomposition of (72), with (19), (23), and (73), reads

$$h_{\mathrm{T}} \, \mathcal{H} \wedge d\sigma + h_{\mathrm{S}} \, \mathcal{D} = -\lambda \left[(-1)^{s_0} f_{\mathrm{T}} \frac{1}{c} \underline{*} E - f_{\mathrm{S}} \, c \, d\sigma \wedge \underline{*} B \right] \tag{74}$$

or, equivalently,

$$E = \frac{c}{\lambda} (-1)^{s_0+s_1+1} \frac{h_{\mathrm{S}}}{f_{\mathrm{T}}} \underline{*} \mathcal{D} \,, \qquad B = \frac{1}{\lambda c} (-1)^{s_1+1} \frac{h_{\mathrm{T}}}{f_{\mathrm{S}}} \underline{*} \mathcal{H} \,. \tag{75}$$

Hence instead of the four 3-dimensional forms $E, \mathcal{D}, \mathcal{H}, B$, we can now consider only one pair of forms. We choose the excitation $H = (\mathcal{H}, \mathcal{D})$ since H is straightforwardly determined by its sources ρ and j.

5 Positivity of the electromagnetic energy

Let us now rewrite the "time" $(t = \sigma)$ component Σ_0 in term of the excitation $H = (\mathcal{H}, \mathcal{D})$. Substituting (75) in (43) we obtain

$$\begin{aligned}
\Sigma_0 = &-\frac{c}{\lambda} (-1)^{s_0+s_1} h_{\mathrm{T}} h_{\mathrm{S}} \underline{*} \, \mathcal{D} \wedge \mathcal{H} \wedge d\sigma \\
&+\frac{c}{2\lambda} (-1)^{s_0+s_1+1} \underline{*} \, \mathcal{D} \wedge \mathcal{D} + \frac{1}{2\lambda c} (-1)^{s_1} \underline{*} \, \mathcal{H} \wedge \mathcal{H} \,.
\end{aligned} \tag{76}$$

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One should compare this expression with the electric current (17). We recognize in the first term the energy flux density (or Poynting) 2-form

$$s := \frac{c}{\lambda} (-1)^{s_0+s_1} h_{\mathrm{T}} h_{\mathrm{S}} \underline{*} \mathcal{D} \wedge \mathcal{H} \,. \tag{77}$$

The remaining terms in (76) represent the energy density 3-forms of the electric and magnetic fields, respectively:

$$u_{\mathrm{el}} := \frac{c}{2\lambda} (-1)^{s_0+s_1+1} \underline{*} \mathcal{D} \wedge \mathcal{D} \,, \qquad u_{\mathrm{mg}} := \frac{1}{2\lambda c} (-1)^{s_1} \underline{*} \mathcal{H} \wedge \mathcal{H} \,. \tag{78}$$

Thus (76) can be rewritten as

$$\Sigma_0 = -s \wedge d\sigma + u_{\mathrm{el}} + u_{\mathrm{mg}} \,. \tag{79}$$

Consequently, the relation $d\Sigma_0 = 0$ yields the standard form of the continuity equation for the electromagnetic energy

$$\underline{d} s + \frac{\partial}{\partial \sigma} (u_{\mathrm{el}} + u_{\mathrm{mg}}) = 0 \,. \tag{80}$$

Let us calculate the 3-form $\underline{*}\mathcal{H} \wedge \mathcal{H}$ in local coordinates. We decompose the 1-form $\mathcal{H}$ according to

$$\mathcal{H} = \mathcal{H}_1 dx + \mathcal{H}_2 dy + \mathcal{H}_3 dz \,. \tag{81}$$

Thus,

$$\underline{*}\mathcal{H} = \mathcal{H}_1 (-1)^{s_1} dy \wedge dz - \mathcal{H}_2 dx \wedge dz + \mathcal{H}_3 dx \wedge dy \,, \tag{82}$$

and, consequently,

$$\underline{*}\mathcal{H} \wedge \mathcal{H} = \left[(-1)^{s_1} (\mathcal{H}_1)^2 + (\mathcal{H}_2)^2 + (\mathcal{H}_3)^2 \right] {}^{(3)}\mathrm{vol} \,. \tag{83}$$

Analogously, we decompose the 2-form $\mathcal{D}$ according to

$$\mathcal{D} = \mathcal{D}^1 dy \wedge dz + \mathcal{D}^2 dz \wedge dx + \mathcal{D}^3 dx \wedge dy \tag{84}$$

and obtain

$$\underline{*}\mathcal{D} \wedge \mathcal{D} = (-1)^{s_1} \left[(-1)^{s_1} (\mathcal{D}^1)^2 + (\mathcal{D}^2)^2 + (\mathcal{D}^3)^2 \right] {}^{(3)}\mathrm{vol} \,. \tag{85}$$

We substitute (83) and (85) in (78) and find

$$u_{\text{el}} = \frac{(-1)^{s_0+1} c}{2\lambda} \left[(-1)^{s_1} (\mathcal{D}^1)^2 + (\mathcal{D}^2)^2 + (\mathcal{D}^3)^2 \right] {}^{(3)}\text{vol}\,, \tag{86}$$

$$u_{\text{mg}} = \frac{(-1)^{s_1}}{2\lambda c} \left[(-1)^{s_1} (\mathcal{H}_1)^2 + (\mathcal{H}_2)^2 + (\mathcal{H}_3)^2 \right] {}^{(3)}\text{vol}\,. \tag{87}$$

Incidentally, in SI notation $\varepsilon_0 = \lambda/c$ and $\mu_0 = 1/(\lambda c)$. Hence (86) and (87) have, indeed, the correct dimensions of energies.

For Euclidean signature with $s_0 = s_1 = 0$, we have the electric energy density

$$u_{\text{el}} = -\frac{c}{2\lambda} \left[(\mathcal{D}^1)^2 + (\mathcal{D}^2)^2 + (\mathcal{D}^3)^2 \right] {}^{(3)}\text{vol} \tag{88}$$

and the magnetic energy density

$$u_{\text{mg}} = \frac{1}{2\lambda c} \left[(\mathcal{H}_1)^2 + (\mathcal{H}_2)^2 + (\mathcal{H}_3)^2 \right] {}^{(3)}\text{vol}\,. \tag{89}$$

Hence in the Euclidean case, the electric and magnetic energy densities are of opposite sign. This agrees with the result of Brill [2].

For Minkowskian signature with $s_0 = 1, s_1 = 0$, the electric and magnetic energy densities are

$$u_{\text{el}} = \frac{c}{2\lambda} \left[(\mathcal{D}^1)^2 + (\mathcal{D}^2)^2 + (\mathcal{D}^3)^2 \right] {}^{(3)}\text{vol} \tag{90}$$

and

$$u_{\text{mg}} = \frac{1}{2\lambda c} \left[(\mathcal{H}_1)^2 + (\mathcal{H}_2)^2 + (\mathcal{H}_3)^2 \right] {}^{(3)}\text{vol}\,, \tag{91}$$

respectively. Thus, in the Minkowskian case, the electric and magnetic energy densities are positive provided $\lambda > 0$.

For different signatures, the Poynting 2-form s has different signs and even depends on the value of the factor $h_{\text{T}} h_{\text{S}}$. This does not prevent the conservation law for the total electromagnetic energy to be formally valid for all signatures. Indeed, it is a consequence of the four dimensional conservation law that holds independently of the value of the factors h_{T} and h_{S}. Nevertheless, as we will see below, for Minkowskian signature, s will finally have its correct form $s = (c/\lambda) \underline{*}\mathcal{D} \wedge \mathcal{H} = E \wedge \mathcal{H}$.

6 Field equations — derivation of factors

The first pair of Maxwell equations (21) is already given in term of the pair $(\mathcal{H}, \mathcal{D})$:

$$h_{\mathrm{T}}\, \underline{d}\, \mathcal{H} + h_{\mathrm{S}}\, \frac{\partial}{\partial \sigma} \mathcal{D} = i_{\mathrm{T}} j\,, \qquad h_{\mathrm{S}}\, \underline{d}\, \mathcal{D} = -i_{\mathrm{T}} \rho\,. \qquad (92)$$

We substitute (75) in the second pair of Maxwell equations (29) and obtain

$$(-1)^{s_0} h_{\mathrm{S}}\, c\, \underline{d}\, (\underline{*}\, \mathcal{D}) + h_{\mathrm{T}}\, \frac{1}{c}\, \frac{\partial}{\partial \sigma} (\underline{*}\, \mathcal{H}) = 0\,, \qquad \underline{d}\, (\underline{*}\, \mathcal{H}) = 0\,. \qquad (93)$$

The system of these four equations determines completely $\mathcal{H}$ and $\mathcal{D}$ for pre-scribed sources j and ρ. Consequently, the fact that the factors f_{T} and f_{S} are no longer involved in the field equations (92) and (93) means that these constants have to be treated as conventional. It is natural to accept the usual convention, i.e., to require E to be in the same direction as $\underline{*}\, \mathcal{D}$ and similarly for B and $\underline{*}\, \mathcal{H}$. Thus, from (75), we have

$$f_{\mathrm{T}} = (-1)^{s_0 + s_1 + 1} h_{\mathrm{S}}\,, \qquad f_{\mathrm{S}} = (-1)^{s_1 + 1} h_{\mathrm{T}}\,, \qquad \text{for} \qquad \lambda > 0\,. \qquad (94)$$

and

$$f_{\mathrm{T}} = (-1)^{s_0 + s_1} h_{\mathrm{S}}\,, \qquad f_{\mathrm{S}} = (-1)^{s_1} h_{\mathrm{T}}\,, \qquad \text{for} \qquad \lambda < 0\,. \qquad (95)$$

Observe another property of the system (92), (93). The constant i_{T} appears only as a factor in front of the charge ρ and the current j. We take into account that the 2-form j always represents the current density of some charge. Thus, even if we define the 3-form ρ to be positive, the charge density $i_{\mathrm{T}}\rho$ can be of either sign. Thus we find: *Electric charges can be of two types: positive and negative charges.* This result is valid for all signatures. Incidentally, in an analogous treatment of gravity, one finds only one type of "charge", namely mass-energy with a positive sign; in contrast, negative mass-energy doesn't exist.

Now we can absorb the factor i_{T} into ρ and j or, in other words, put $i_{\mathrm{T}} = 1$. Consequently ρ can carry two opposite signs and j two opposite directions. We rewrite the system (92), (93) as

$$\underline{d}\, \mathcal{H} + h_{\mathrm{T}} h_{\mathrm{S}}\, \frac{\partial}{\partial \sigma} \mathcal{D} = h_{\mathrm{T}} j\,, \qquad \underline{d}\, \mathcal{D} = -h_{\mathrm{S}}\, \rho\,, \qquad (96)$$

and

$$c\,\underline{d}\,(*\,\mathcal{D}) + (-1)^{s_0} h_\mathrm{T} h_\mathrm{S} \frac{1}{c} \frac{\partial}{\partial\sigma}(*\,\mathcal{H}) = 0\,, \qquad \underline{d}\,(*\,\mathcal{H}) = 0\,. \tag{97}$$

The relative signs on the left-hand-sides of $(96)_1$ and $(97)_1$, respectively, depend only on s_0 and not on $h_\mathrm{T} h_\mathrm{S}$. For $s_0 = 1$, the relative signs in $(96)_1$ and $(97)_1$ are opposite, in accordance with the Lenz rule.

For $s_0 = 0$, we have the same pair of signs — the anti-Lenz rule. The signature, however, is defined by two factors s_0 and s_1; for instance, $s_0 = 1$, $s_1 = 0$ corresponds to Minkowskian signature as well as $s_0 = 0$, $s_1 = 1$. Thus so far we cannot establish the proper correspondence between the signature and the sign of the induction. It will be done in the next section by applying the expression for the Lorentz force.

At this stage, we are ready to fix the factors h_T and h_S. The electromagnetic excitation H (that is, its projections $\mathcal{H}$ and $\mathcal{D}$) can only be measured with the help of the current J. Thus, in $(96)_2$, the factor h_S fixes the direction that we ascribe to the field $\mathcal{D}$ if it emanates, say, from a positive charge. Conventionally, $\mathcal{D}$ is defined such that it is directed from positive to negative charges. Thus $h_\mathrm{S} = -1$ and (96) reads

$$\underline{d}\,\mathcal{H} = h_\mathrm{T}\left(j + \dot{\mathcal{D}}\right)\,, \qquad \underline{d}\,\mathcal{D} = \rho\,. \tag{98}$$

Eq.(98) represents the Ampère-Maxwell law. Note that the contributions of j and $\dot{\mathcal{D}}$ already emerge with the correct relative sign. In analogy to $\mathcal{D}$, the magnetic excitation $\mathcal{H}$ is defined such that it has $j + \dot{\mathcal{D}}$ as source, that is, $h_\mathrm{T} = 1$. Accordingly, the Maxwell equations (96), (97) finally read

$$\underline{d}\,\mathcal{H} - \frac{\partial}{\partial\sigma}\mathcal{D} = j\,, \qquad \underline{d}\,\mathcal{D} = \rho\,, \tag{99}$$

and

$$c\,\underline{d}\,(*\,\mathcal{D}) + (-1)^{s_0} \frac{1}{c} \frac{\partial}{\partial\sigma}(*\,\mathcal{H}) = 0\,, \qquad \underline{d}\,(*\,\mathcal{H}) = 0\,. \tag{100}$$

We collect the sign factors the values of which have already been set:

$$i_\mathrm{T} = 1\,, \quad i_\mathrm{S} = -1\,, \quad h_\mathrm{T} = 1\,, \quad h_\mathrm{S} = -1\,. \tag{101}$$

The sign factors of the field strength F, in our convention (94), depend on the signature *and* on the sign of λ. For $\lambda > 0$, we listed the values for the different signatures in Table 2.

Table 2. The sign factors f_{T}, f_{S} of the field strength F for different signatures in the case of $\lambda > 0$

Signature	f_{T}	f_{S}
Minkowski (aka Lorentz)	-1	-1
Euclid	1	-1
$(-,-,+,+)$	1	1

7   Lorentz force — relation between different signatures

The components of the Lorentz force density (24) and (25), for $i_{\mathsf{T}} = 1$, see (101), can be rewritten according to

$$f_0 = -f_{\mathsf{T}}\, E \wedge j \wedge d\sigma\,, \qquad f_\mu = \Big[f_{\mathsf{T}}(e_\mu \rfloor E)\rho + f_{\mathsf{S}}(e_\mu \rfloor B) \wedge j \Big] \wedge d\sigma\,. \quad (102)$$

We substitute (75) in (102) and use the already fixed values (101) of the factors $h_{\mathsf{T}}, h_{\mathsf{S}}$, and i_{T}. Then,

$$f_0 = -\frac{c}{\lambda}\,(-1)^{s_0+s_1}\, \underset{*}{*}\,\mathcal{D} \wedge j \wedge d\sigma \tag{103}$$

and

$$f_\mu = \frac{1}{\lambda}\Big[(-1)^{s_0+s_1} c\,(e_\mu \rfloor \underset{*}{*}\,\mathcal{D})\rho + (-1)^{s_1+1} \frac{1}{c}\,(e_\mu \rfloor \underset{*}{*}\,\mathcal{H}) \wedge j \Big] \wedge d\sigma\,. \tag{104}$$

The latter expression represents the ordinary electromagnetic 3-force. Using the identity $e_\mu \rfloor * w = *(w \wedge dx_\mu)$, we can rewrite it as

$$f_\mu = \frac{1}{\lambda}\Big[(-1)^{s_0+s_1} c\,\underset{*}{*}\,\rho\,\mathcal{D} + (-1)^{s_1} \frac{1}{c}\,\underset{*}{*}\,j \wedge \mathcal{H} \Big] \wedge dx_\mu \wedge d\sigma\,. \tag{105}$$

Let us examine this expression for the different signatures. For static fields $\mathcal{D}$ and $\mathcal{H}$, the signature is completely eliminated from the field equations (99), (100). Thus, for given sources ρ and j, we have the same static fields for all signatures.

● For $s_0 = 1$, $s_1 = 0$, we have the ordinary Maxwell-Lorentz electrodynamics in Riemannian spacetime with Minkowskian signature. The first term on the right-hand-side of (105) represents the electric force between two static charges. In particular, it yields attraction between opposite charges and repulsion between charges of the same sign: this is Dufay's law. The second term

describes the magnetic force. In particular, it is responsible for the pulling of a ferromagnetic core into a solenoid independently on the direction of the current, in accordance with Lenz's rule.

• For $s_0 = 0$, $s_1 = 0$, we deal with Euclidean electrodynamics. In contrast to the case of Minkowskian signature, the energy density of the electromagnetic field does not have a definite sign. Hence also the sign of λ is not defined.

In the case $\lambda > 0$, the Euclidean electric term in (105) is opposite to the corresponding term of ordinary (Minkowskian) electrodynamics. Consequently, we obtain an anti-Dufay law: opposite charges repel whereas charges of the same sign attract each other. As for the Euclidean magnetic force, it comes with the same sign as in ordinary electrodynamics, in accordance with the Lenz rule.

In the case $\lambda < 0$, the situation is opposite: the Dufay law for charges and the anti-Lenz rule for currents.

These results are in correspondence with the signs of the Euclidean electric and magnetic energy densities (88,89). They are partly presented in Brill's analysis [2].

• For $s_0 = 1$, $s_1 = 1$, i.e., for the signature $(-,-,+,+)$, the two factors in (105) have opposite signs relative to the Minkowskian case. Thus, for $\lambda > 0$, we have anti-Dufay and anti-Lenz laws. For $\lambda < 0$, the laws are the same as in ordinary electrodynamics. A surprising fact is that the forces have definite signs, although the signs of the energy densities are undefined.

8   Main results and discussion

Let us recall the main points of our analysis:

 (i) We formulate, in a metric-free form, the field equations, the Lorentz force, and the conservation law for energy-momentum.
 (ii) The spacetime relation is assumed to be local, linear, and symmetric.
(iii) The field equations and the Lorentz force are shown to be compatible with a certain energy-momentum current Σ_α.
(iv) In order to provide a physical interpretation of the 4-dimensional quantities, we construct their $(1+3)$-decompositions with a number of free sign factors.
 (v) We observe a specific symmetry of the energy-momentum current Σ_α, namely electric/magnetic reciprocity. In $(1+3)$-decomposition it translates into an exchange between electric and magnetic fields.
(vi) If electric/magnetic reciprocity is applied to the spacetime relation, see (ii), it yields a metric of Lorentzian type.

(vii) For all possible signatures of the 4-dimensional metric, we obtain the expressions for the electric and magnetic energy densities. The metric of a Lorentzian type turns out to be related to a positive electromagnetic energy density. This result does not depend on the values of the sign factors.

(viii) We analyze the $(1 + 3)$-decompositions of the field equations. We derive which sign factors are conventional and which do depend on the signature. We find that the electric charge has two possible signs for all signatures.

(ix) For all signatures, we derive the features of the interactions between charges (Dufay's law) and between currents (Lenz's rule).

All in all, our discussion has shown that the *Lorentzian* signature of the metric of spacetime originates in properties of the electromagnetic spacetime relation $H = H(F)$. If the spacetime relation is local, linear, and symmetric, the requirement of electric/magnetic reciprocity induces the lightcone (with Lorentzian signature) and a positive definite electromagnetic energy together with Dufay's law and Lenz's rule.

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